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**ON THE EQUIVALENCE PROBLEM FOR MANIFOLDS
OF INDEFINITE METRIC**

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Conditions, related to the Kulkarni's equivalence problem are considered for indefinite Riemannian and Kaehlerian manifolds. Corresponding theorems are obtained for the Ricci curvatures as well as for the holomorphic sectional curvatures of indefinite Kaehler manifolds.

1. Let M be a Riemannian manifold of definite or indefinite metric g . A 2-plane α in a tangent space is said to be nondegenerate, weakly degenerate or strongly degenerate, if the rank of the restriction of g on α is 2, 1 or 0, respectively. A vector ξ on M is said to be isotropic, if $g(\xi, \xi) = 0$ and $\xi \neq 0$. Of course, for degenerate planes and isotropic vectors one speaks only when the metric is indefinite. The curvature of a nondegenerate 2-plane α with a basis $\{x, y\}$ is defined by

$$K(\alpha) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where R denotes the curvature tensor and

$$\pi_1(z, u, v, w) = g(z, w)g(u, v) - g(z, v)g(u, w).$$

Let \overline{M} be another Riemannian manifold of definite or indefinite metric. The corresponding objects for \overline{M} will be denoted by a bar overhead. A diffeomorphism f of M onto \overline{M} is said to be sectional curvature preserving [7], if

$$(1.1) \quad \overline{K}(f_*\alpha) = K(\alpha)$$

for every nondegenerate 2-plane α on M . In [7] R. S. Kulkarni investigates the converse of the so-called "theorema egregium" of Gauss and proves that any sectional curvature preserving diffeomorphism of Riemannian manifolds of nowhere constant sectional curvature and dimension ≥ 4 is trivial, i.e. it is an isometry. The related questions for the Ricci curvature and for the holomorphic sectional curvature of Kaehler manifolds are considered in [8], [9]. Sectional curvature preserving diffeomorphisms of indefinite Riemannian manifolds are studied in [10]. The condition, corresponding to (1.1) for degenerate planes is

$$(1.2) \quad \lim_{\alpha \rightarrow \alpha_0} \frac{\overline{K}(f_*\alpha)}{K(\alpha)} = 1,$$

where the degenerate 2-plane α_0 is approximated by nondegenerate 2-planes, whose images are also nondegenerate. In [6] we examine diffeomorphisms, satisfying (1.2) for weakly or for strongly degenerate 2-planes. Then they appear the manifolds of quasi-constant curvature and the manifolds of recurrent curvature in the sense of Walker, whose definitions we recall:

An n -dimensional (indefinite) Riemannian manifold is said to be a K_n^* -manifold [11], if either it is recurrent (i.e. $\nabla R = \alpha \otimes R$, where $\alpha \neq 0$) or it is symmetric and there exists a differential form $\alpha \neq 0$, such that

$$\sum_{cycl\ x,y,z} \alpha(x)R(y,z,u,v) = 0 .$$

Walker [11] showed, that α is defined by $\alpha(X) = g(\nabla v, X)$, where v is a smooth function (called recurrence-function) and ∇v denotes the gradient of v .

An n -dimensional (indefinite) Riemannian manifold is said to be of quasi-constant curvature [1], [2], if it is conformally flat and there exist functions H, N and a unit vector V , such that the curvature tensor has the form

$$R = (N - H)\varphi(B) + H\pi_1 ,$$

where $B(X, Y) = g(X, V)g(Y, V)$ and φ is defined by

$$\varphi(Q)(x, y, z, u) = g(x, u)Q(y, z) - g(x, z)Q(y, u) + g(y, z)Q(x, u) - g(y, u)Q(x, z)$$

for any symmetric tensor Q of type (0,2). Such a manifold we shall denote by $M(H, N, V)$.

In this paper we consider diffeomorphisms of indefinite Riemannian manifolds, satisfying conditions, analogous to (1.2) for the Ricci curvature and for the holomorphic sectional curvature of indefinite Kaehler manifolds. We need the following Lemma.

Lemma [6]. Let f be a diffeomorphism of indefinite Riemannian manifolds of dimension $n \geq 3$. Let in a point $p \in M$ there exists an isotropic vector ξ , such that every isotropic vector which is sufficiently close to ξ is mapped by f_* in an isotropic vector in $f(p)$. Then f is a homothety in p .

2. Let us recall that the Ricci curvature in the direction of a nonzero nonisotropic vector x is defined by

$$K_S(x) = \frac{S(x, x)}{g(x, x)} ,$$

where S is the Ricci tensor of M . As analogue of the Ricci curvature preserving diffeomorphisms [8], it is natural to consider diffeomorphisms, satisfying

$$(2.1) \quad \lim_{x \rightarrow \xi} \frac{\overline{K_S}(f_*x)}{K_S(x)} = 1 ,$$

when the isotropic vector ξ is approximated by nonisotropic vectors, whose images are also nonisotropic. Then we have

Theorem 1. Let M and \overline{M} be indefinite Riemannian manifolds of dimension $n \geq 3$ and let f be a diffeomorphism of M onto \overline{M} satisfying (2.1) for every isotropic vector

ξ on M . If M is nowhere Einsteinian, then f is conformal. Let $f^*\bar{g} = e^{2\sigma}g$ and assume that M is conformally flat. Then:

a) if $\nabla\sigma$ vanishes identically, then f is an isometry;
b) if $\nabla\sigma$ is isotropic, then M is a conformal flat K_n^* -space and σ is a function of the recurrence-function;

c) if $\|\nabla\sigma\|^2$ doesn't vanish, then M is a manifold $M(H, N, \nabla\sigma/\|\nabla\sigma\|)$ of quasi-constant curvature, ∇H and ∇N being proportional to $\nabla\sigma$.

Proof. Let M be non-Einsteinian in p , i.e. the Ricci tensor S_p is not proportional to g_p . Then there exists an isotropic vector ξ in p , such that $S(\xi, \xi) \neq 0$ [4]. By continuity $S(\xi', \xi') \neq 0$ for every isotropic vector ξ' , sufficiently close to ξ . Then from (2.1) it follows that every such vector is mapped by f_* onto an isotropic one. By the Lemma f is a homothety in p . Since the set of points in which M is not Einsteinian is dense, then f is conformal.

Let $f^*\bar{g} = e^{2\sigma}g$. For the sake of simplicity of the denotations we identify M with \bar{M} via f and omit f_* . Then (2.1) implies $\bar{S}(\xi, \xi) = e^{2\sigma}S(\xi, \xi)$. Hence it follows (see [4])

$$(2.2) \quad \bar{S} = e^{2\sigma} \left\{ S + \frac{\bar{\tau} - \tau}{n} g \right\},$$

where τ denotes the scalar curvature of M . Since (M, \bar{g}) and (M, g) are conformally flat, their Weil conformal curvature tensors vanish [5], i.e.

$$(2.3) \quad \begin{aligned} \bar{R} &= \frac{1}{n-2} \varphi(\bar{S}) - \frac{\bar{\tau}}{(n-1)(n-2)} \bar{\pi}_1, \\ R &= \frac{1}{n-2} \varphi(S) - \frac{\tau}{(n-1)(n-2)} \pi_1. \end{aligned}$$

From (2.2) and (2.3) it follows

$$\bar{R} = e^{4\sigma} \left\{ R + \frac{\bar{\tau} - \tau}{n(n-1)} \pi_1 \right\}.$$

Hence (1.2) is satisfied and the rest of the theorem follows from Theorem 2 in [6].

3. Let M be an indefinite Kaehler manifold with metric g and almost complex structure J . A 2-plane α is said to be holomorphic, if $\alpha = J\alpha$. Note that a degenerate holomorphic 2-plane is necessarily strongly degenerate. The Bochner curvature tensor B for M is defined by

$$B = R - \frac{1}{2(n+2)}(\varphi + \psi)(S) + \frac{\tau}{4(n+1)(n+2)}(\pi_1 + \pi_2),$$

where $2n$ is the dimension of M ,

$$\begin{aligned} \psi(Q)(x, y, z, u) &= g(x, Ju)Q(y, Jz) - g(x, Jz)Q(y, Ju) - 2g(x, Jy)Q(z, Ju) \\ &\quad + g(y, Jz)Q(x, Ju) - g(y, Ju)Q(x, Jz) - 2g(z, Ju)Q(x, Jy) \end{aligned}$$

for a symmetric tensor Q of type (0,2) and $\pi_2 = \frac{1}{2}\psi(g)$. As usual we denote the curvature of a nondegenerate holomorphic 2-plane with an orthonormal basis $\{x, Jx\}$ by $H(x)$. For

the holomorphic curvatures, i.e. the curvatures of holomorphic planes, the analogue of (1.2) is

$$(3.1) \quad \lim_{x \rightarrow \xi} \frac{\overline{H}(f_*x)}{H(x)} = 1 ,$$

where the isotropic vector ξ is approximated by nonisotropic vectors, whose images are also nonisotropic and with the natural requirement that the image of any holomorphic 2-plane is also holomorphic (see also [9]). Then we have

Theorem 2. Let M and \overline{M} be indefinite Kaehler manifolds of dimension $2n \geq 4$ and let f be a diffeomorphism of M onto \overline{M} , satisfying (3.1) for every isotropic vector ξ on M . If M is not of constant holomorphic sectional curvature, then f is a holomorphic or antiholomorphic isometry.

Proof. By Lemma 2 in [9] f is holomorphic or antiholomorphic. Let N be the set of points, in which M is not of constant holomorphic sectional curvature and let $p \in N$. To show that f is a homothety in p , we shall consider two cases:

1) The Bochner curvature tensor of M vanishes in p . Then M cannot be Einsteinian in p , because it is not of constant holomorphic sectional curvature in p . Hence there exists an isotropic vector ξ in $T_p M$, such that $S(\xi, \xi) \neq 0$. Then $S(\xi', \xi') \neq 0$ for every isotropic vector ξ' , sufficiently close to ξ . By (3.1) and $B = 0$ we have

$$\begin{aligned} 1 &= \lim_{x \rightarrow \xi'} \frac{\overline{H}(f_*x)}{H(x)} = \lim_{x \rightarrow \xi'} \frac{\overline{H}(f_*x)g(x, x)}{\frac{4}{n+2}S(x, x) - \frac{\tau}{(n+1)(n+2)}g(x, x)} \\ &= \frac{n+2}{4S(\xi', \xi')} \lim_{x \rightarrow \xi'} \frac{\overline{R}(f_*x, Jf_*x, Jf_*x, f_*x)g(x, x)}{\bar{g}^2(f_*x, f_*x)} . \end{aligned}$$

Hence $\bar{g}(f_*\xi', f_*\xi') = 0$, i.e. $f_*\xi'$ is isotropic. According to the Lemma from section 1 f is a homothety in p .

2) The Bochner curvature tensor doesn't vanish in p . Then there exists an isotropic vector ξ in $T_p M$, such that $R(\xi, J\xi, J\xi, \xi) \neq 0$ [3]. Consequently $R(\xi', J\xi', J\xi', \xi') \neq 0$ for every isotropic vector ξ' , sufficiently close to ξ . Then

$$1 = \lim_{x \rightarrow \xi'} \frac{\overline{H}(f_*x)}{H(x)} = \frac{1}{R(\xi', J\xi', J\xi', \xi')} \lim_{x \rightarrow \xi'} \frac{\overline{R}(f_*x, Jf_*x, Jf_*x, f_*x)}{\bar{g}^2(f_*x, f_*x)} g^2(x, x)$$

which implies again $\bar{g}(f_*\xi', f_*\xi') = 0$ and hence f is a homothety in p .

So f is a homothety in p . Since M is not of constant holomorphic sectional curvature, the set N is dense in M . This implies that f is conformal, i.e. $f^*g = \lambda g$, where λ is a smooth function. Denote by Φ the fundamental form of M : $\Phi(X, Y) = g(JX, Y)$. Then $f^*\Phi = \lambda\Phi$. Since Φ and $\overline{\Phi}$ are closed, this yields $\alpha\lambda \wedge \Phi = 0$ (see also [9]), which implies that λ is a (nonzero) constant, thus proving the theorem.

Corollary. If in Theorem 2 M has nonvanishing Bochner curvature tensor, then f is a holomorphic or antiholomorphic isometry.

Proof. From $f^*\bar{g} = \lambda R$ it follows $f^*\overline{R} = \lambda R$. Hence

$$(3.2) \quad (f^*\overline{R})(\xi, J\xi, J\xi, \xi) = \lambda R(\xi, J\xi, J\xi, \xi) .$$

On the other hand (3.1) implies

$$(3.3) \quad (f^*\overline{R})(\xi, J\xi, J\xi, \xi) = \lambda^2 R(\xi, J\xi, J\xi, \xi)$$

for every isotropic vector ξ on M . Since M has nonvanishing Bochner curvature tensor, there exists a point p in M and an isotropic vector ξ in $T_p M$, such that $R(\xi, J\xi, J\xi, \xi) \neq 0$ [3]. Then (3.2) and (3.3) yield $\lambda = 1$, proving the assertion.

Note that if (2.1) is satisfied for every isotropic vector for indefinite Kaehler manifold and M is not Einsteinian, as in Theorem 1 we obtain $f^*\overline{g} = \lambda g$ for $\lambda \in \mathfrak{R}M$. Assume that f is holomorphic or antiholomorphic. Then as in Theorem 2 λ is a constant and similarly to the case in the Corollary $\lambda = 1$. Thus we have

Theorem 3. Let M and \overline{M} be indefinite Kaehler manifolds of dimension $2n \geq 4$ and let f be a holomorphic or antiholomorphic diffeomorphism of M onto \overline{M} , satisfying (2.1) for every isotropic vector ξ on M . If M is not Einsteinian, then f is an isometry.

The same assertion holds for diffeomorphisms of definite or indefinite Kaehler manifolds if (2.1) is changed by

$$\overline{K}_{\overline{S}}(f_*x) = K_S(x)$$

for every unit vector x on M .

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